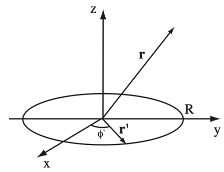
Exercise sheet #6

Problem 1. A circular ring in the xy plane (radius R, centered at the origin) carries a uniform line charge λ. Find the first three terms (n=0,1,2) in the multipole expansion of $V(r,\theta)$



Solution: For a line charge, $\rho(\mathbf{r}')\,d\tau' \to \lambda(\mathbf{r}')\,dl'$, which in this case becomes $\lambda R d\phi'$.

$\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$,
$\mathbf{r}' = R\cos\phi'\hat{\mathbf{x}} + R\sin\phi'\hat{\mathbf{y}}$, so
$\mathbf{r}\cdot\mathbf{r}' = rR\sin\theta\cos\phi\cos\phi' + rR\sin\theta\sin\phi\sin\phi' = rR\cos\alpha$,
$\cos\alpha = \sin\theta\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right)$.

$n = 0$:

$$\int \rho\left(\mathbf{r}'\right) d\tau' \to \lambda R\int_0^{2\pi} d\phi' = 2\pi R\lambda; \quad V_0 = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda}{r} = \frac{\lambda}{2\epsilon_0}\frac{R}{r}.$$

$n = 1$:

$$\int r'\cos\alpha\rho\left(\mathbf{r}'\right) d\tau' \to \int R\cos\alpha\lambda R d\phi' = \lambda R^2\sin\theta\int_0^{2\pi}\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right) d\phi' = 0; V_1 = 0.$$

$n = 2$:

$$\int \left(r'\right)^2 P_2(\cos\alpha)\rho\left(\mathbf{r}'\right) d\tau' \to \int R^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\lambda R d\phi' = \frac{\lambda R^3}{2}\int\left[3\sin^2\theta\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right)^2 - 1\right] d\phi'$$

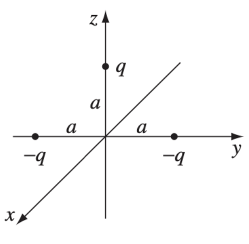
$$= \frac{\lambda R^3}{2}\left[3\sin^2\theta\left(\cos^2\phi\int_0^{2\pi}\cos^2\phi' d\phi' + \sin^2\phi\int_0^{2\pi}\sin^2\phi' d\phi' + 2\sin\phi\cos\phi\int_0^{2\pi}\sin\phi'\cos\phi' d\phi'\right) - \int_0^{2\pi} d\phi'\right]$$

$$= \frac{\lambda R^3}{2}\left[3\sin^2\theta\left(\pi\cos^2\phi + \pi\sin^2\phi + 0\right) - 2\pi\right] = \frac{\pi\lambda R^3}{2}\left(3\sin^2\theta - 2\right) = -\pi\lambda R^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

$$\text{So:} \quad V_2 = -\frac{\lambda}{8\epsilon_0}\frac{R^3}{r^3}\left(3\cos^2\theta - 1\right) = -\frac{\lambda}{4\epsilon_0}\frac{R^3}{r^3}P_2(\cos\theta).$$

□

Problem 2. Three point charges are located as shown in the figure below, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.

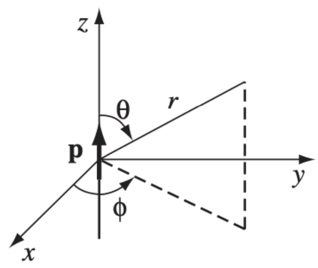


Solution: $Q = -q$, so $V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$; $\mathbf{p} = qa\hat{\mathbf{z}}$, so $V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$. Therefore

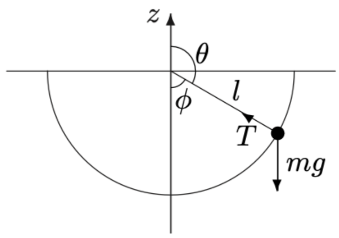
$$V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right). \quad \mathbf{E}(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\hat{\mathbf{r}} + \frac{a}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\right]$$

□

Problem 3. An ideal electric dipole is situated at the origin, and points in the z direction, as in the figure below. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin



Solution. :



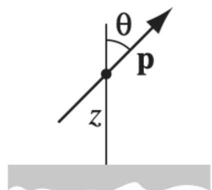
$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

Now consider the pendulum: $\mathbf{F} = -mg\hat{\mathbf{z}} - T\hat{\mathbf{r}}$, where $T - mg\cos\phi = mv^2/l$ and (by conservation of energy) $mgl\cos\phi = (1/2)mv^2 \Rightarrow v^2 = 2gl\cos\phi$ (assuming it started from rest at $\phi = 90°$, as stipulated). But $\cos\phi = -\cos\theta$, so $T = mg(-\cos\theta) + (m/l)(-2gl\cos\theta) = -3mg\cos\theta$, and hence

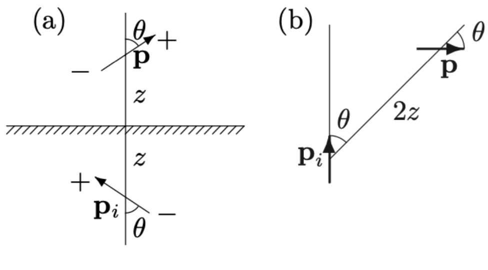
$$\mathbf{F} = -mg(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) + 3mg\cos\theta\hat{\mathbf{r}} = mg(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

This total force is such as to keep the pendulum on a circular arc, and it is identical to the force on q in the field of a dipole, with $mg \leftrightarrow qp/4\pi\epsilon_0 l^3$. Evidently q also executes semicircular motion, as though it were on a tether of fixed length l. □

Problem 4. A perfect dipole $\mathbf{p}$ is situated at a distance z above an infinite conducting grounding plane (See figure below). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on $\mathbf{p}$. If the dipole is free to rotate, in what orientation will it come to rest?



Solution: Use image dipole as shown in Fig. (a) below . Redraw, placing $\mathbf{p}_i$ at the origin, Fig. (b).



$$\mathbf{E}_i = \tfrac{p}{4\pi\epsilon_0(2z)^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}); \quad \mathbf{p} = p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}}$$

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}_i = \frac{p}{4\pi\epsilon_0(2z)^3}(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \times (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\Big]$$

$$= \frac{p^2}{4\pi\epsilon_0(2z)^3}[\cos\theta\sin\theta\hat{\boldsymbol{\phi}} + 2\sin\theta\cos\theta(-\hat{\boldsymbol{\phi}})]$$

where q is the charge of the electron

$$= \frac{p^2\sin\theta\cos\theta}{4\pi\epsilon_0(2z)^3}(-\hat{\boldsymbol{\phi}}) \quad \text{(out of the page).}$$

and a is the Bohr radius. Find the atomic polarizability of such an atom. [Hint: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$ □

Problem 5. According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3}e^{-2r/a}$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [Hint: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$

Solution: First find the field, at radius r, using Gauss' law: $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$, or $E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}Q_{\text{enc}}$.

$$Q_{\text{enc}} = \int_0^r \rho d\tau = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2\bar{r}/a}\bar{r}^2 d\bar{r} = \frac{4q}{a^3}\left[-\frac{a}{2}e^{-2\bar{r}/a}\left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2}\right)\right]\Bigg|_0^r$$

$$= -\frac{2q}{a^2}\left[e^{-2r/a}\left(r^2 + ar + \frac{a^2}{2}\right) - \frac{a^2}{2}\right] = q\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

[Note: $Q_{\text{enc}}(r\to\infty) = q$.] So the field of the electron cloud is $E_{\text{e}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$. The proton will be shifted from $r = 0$ to the point d where $E_{\text{e}} = E$ (the external field):

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right].$$

Expanding in powers of (d/a) :

$$e^{-2d/a} = 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3!}\left(\frac{2d}{a}\right)^3 + \cdots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots$$

$$1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) = 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)$$

$$= 1 - 1 - 2\frac{d}{a} - 2\frac{d^2}{a^2} + 2\frac{d}{a} + 4\frac{d^2}{a^2} + 4\frac{d^3}{a^3} - 2\frac{d^2}{a^2} - 4\frac{d^3}{a^3} + \frac{4}{3}\frac{d^3}{a^3} + \cdots$$

$$= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{ higher order terms.}$$

$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right) = \frac{1}{4\pi\epsilon_0}\frac{4}{3a^3}(qd) = \frac{1}{3\pi\epsilon_0 a^3}p. \quad \alpha = 3\pi\epsilon_0 a^3.$ □